

Distinct distances between a collinear set and an arbitrary set of points*

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Abstract

We consider the number of distinct distances between two finite sets of points in \mathbb{R}^k , for any constant dimension $k \geq 2$, where one set P_1 consists of n points on a line l , and the other set P_2 consists of m arbitrary points, such that no hyperplane orthogonal to l and no hypercylinder having l as its axis contains more than $O(1)$ points of P_2 . The number of distinct distances between P_1 and P_2 is then

$$\Omega \left(\min \left\{ n^{2/3} m^{2/3}, \frac{n^{10/11} m^{4/11}}{\log^{2/11} m}, n^2, m^2 \right\} \right).$$

Without the assumption on P_2 , there exist sets P_1, P_2 as above, with only $O(m+n)$ distinct distances between them.

1 Introduction

The problem of bounding the minimum number of distinct distances that can be determined by a set of n points in the plane was first proposed by Paul Erdős in 1946 [7]. The problem was stated as follows:

“Let $[P_n]$ be the class of all planar subsets P_n of n points and denote by $f(n)$ the minimum number of different distances determined by its n points for P_n an element of $[P_n]$.”

Erdős then proved the bounds $\sqrt{n-3/4} - 1/2 \leq f(n) \leq cn/\sqrt{\log n}$, stating before the proof:

“The following theorem establishes rough bounds for arbitrary n . Though I have sought to improve this result for many years, I have not been able to do so.”

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Erdős conjectured that the actual value of $f(n)$ is much closer to the upper bound. Several improvements have been made to the lower bound over the years, culminating in the recent bound $f(n) \in \Omega(\frac{n}{\log n})$ of Guth and Katz in 2010 [9]. There still remains a small gap between Erdős's conjectured bound and this best known lower bound. Many interesting variants of the problem are still wide open (that is, with significant gaps between the best known upper and lower bounds), such as the variant considering point sets in \mathbb{R}^d for $d \geq 3$ [7], or variants in which the points are further constrained; e.g., so that no three points are collinear [8].

One such family of problems concern distinct distances between two sets of points, P_1 and P_2 , i.e., the problem is to obtain a lower bound on the cardinality of the set

$$\mathcal{D}(P_1, P_2) = \{\|p - q\| \mid p \in P_1, q \in P_2\}.$$

One of the most basic variants in this family is a planar instance, where P_1 lies on a line l_1 and P_2 lies on a line l_2 , such that l_1 and l_2 are neither parallel nor orthogonal. Purdy conjectured that in this case, for $|P_1| = |P_2| = n$, one always has $\mathcal{D}(P_1, P_2) = \omega(n)$ (e.g., see [2, Section 5.5]). Elekes and Rónyai [6] have proven the conjecture in 2000, without stating a specific bound. A bound of $\Omega(n^{5/4})$ was later given by Elekes [4], and was recently improved by Sharir, Sheffer, and Solymosi to $\Omega(n^{4/3})$ [15]. They have also derived a bound for the unbalanced case, where P_1 and P_2 have different cardinalities.

More recently, Pach and de Zeeuw [12] generalized the result of [15], with the same lower bound, for any pair of sets P_1, P_2 that lie on two respective (possibly coinciding) irreducible plane algebraic curves of constant degree, excluding the cases of parallel lines, orthogonal lines, or concentric circles. In higher dimensions, Charalambides [3] obtained the lower bound $\Omega(n^{5/4})$ for the number of distinct distances between n points on a constant-degree algebraic curve, in any dimension, unless the curve is an *algebraic helix* (see [3] for the definition). The bound has recently been improved to $\Omega(n^{4/3})$ by Raz [13].

In this paper we consider the bipartite distinct distances problem in \mathbb{R}^k , for any fixed dimension $k \geq 2$, involving two sets P_1, P_2 , where P_1 is contained in a line l , and P_2 is fairly arbitrary. Our main result is the following theorem.

Theorem 1 *Let l be a line in \mathbb{R}^k , for any fixed $k \geq 2$, let c be an absolute constant, and let P_1, P_2 be two finite point sets of respective cardinalities n, m , such that $P_1 \subset l$ and P_2 is an arbitrary set of points in \mathbb{R}^k , such that no hyperplane orthogonal to l , and no circular hypercylinder whose axis is l contains more than c points of P_2 . That is, assuming l to be the x_1 -axis,*

$$\begin{aligned} \forall t \in \mathbb{R}, \quad |\{p \in P_2 \mid p_1 = t\}| &\leq c, \quad \text{and} \\ \forall t \in \mathbb{R}, \quad |\{p \in P_2 \mid \rho(p, l) = t\}| &\leq c, \end{aligned}$$

where $\rho(p, l)$ is the (Euclidean) distance of p from l , and p_1 is the first coordinate of p . Then

$$|\mathcal{D}(P_1, P_2)| \in \begin{cases} \Omega(m^2) & \text{if } m \leq n^{1/2} \\ \Omega(n^{2/3}m^{2/3}) & \text{if } n^{1/2} < m \leq n^{4/5}/\log^{3/5} n \\ \Omega\left(\frac{n^{10/11}m^{4/11}}{\log^{2/11} m}\right) & \text{if } n^{4/5}/\log^{3/5} n < m \leq n^3 \\ \Omega(n^2) & \text{if } n^3 < m. \end{cases}$$

The conditions on P_2 are necessary—one can easily construct a set P_1 of n points on a line, and a set P_2 of m points lying in any of the forbidden hyperplanes or hypercylinders, such that $|\mathcal{D}(P_1, P_2)| = O(m + n)$. Such constructions are well known, already for the planar case (e.g., see [15]).

So far, distinct distances in higher dimensions have been studied, within the algebraic framework pioneered by Guth and Katz [9], in only a few papers, including those mentioned above. Although these results do not directly apply to the setup considered in this paper, where only one set is contained in a line, we nevertheless show that this general approach can be adapted to the scenario considered here too. Our bound is worse than that of Pach and de Zeeuw [12], when $|P_2|$ is large relative to $|P_1|$ (we get the same bound for smaller-size sets P_2), but the setup in which it applies is considerably more general, in its assumptions concerning P_2 . Our result can also be compared with Raz’s bound [13], which applies to general algebraic curves in higher dimensions, and is the same as in [12], but it only handles the non-bipartite case, and requires all the points to lie on the curve.

2 Proof of Theorem 1

2.1 Distinct distances in the plane

We first establish Theorem 1 in the plane. The proof in arbitrary dimensions is essentially the same, except that the notations are somewhat more tedious. After handling the planar case, we will comment on the minor modifications that are needed for the general case.

Let l be a line in the plane, which, without loss of generality, we take to be the x -axis. Let P_1 be a set of n points on l , and let P_2 be an arbitrary set of m points, such that no line parallel or orthogonal to l contains more than c points of P_2 , for some (arbitrary) absolute constant c . Define $\mathcal{D}(P_1, P_2)$ as in the introduction, and put $x = |\mathcal{D}(P_1, P_2)|$. To establish the lower bound on x asserted in the theorem, we proceed as follows.

Let P'_2 be a maximal subset of P_2 such that no two points of P'_2 have the same x -coordinate or the same y -coordinate, and all the points of P'_2 lie on the same side of, say, above the x -axis. By the assumptions on P_2 , we have $|P'_2| \geq \lfloor \frac{m}{2(c-1)} \rfloor \in \Theta(m)$. Indeed, we first assume, without loss of generality, that the upper halfplane contains at least $\lfloor m/2 \rfloor$ points of P_2 , and ignore the points lying below l . We then construct P'_2 greedily, picking at each step an arbitrary point from P_2 and discarding the at most $2(c-1)$ other points with the same x -coordinate or the same y -coordinate. We repeat this step until (the subset in the upper halfplane of) P_2 is exhausted, and get a subset P'_2 with all the desired properties. Since $|\mathcal{D}(P_1, P_2)| \geq |\mathcal{D}(P_1, P'_2)|$, a lower bound for the latter quantity will serve as a lower bound for the former as well. To simplify the notation, we continue to refer to the pruned set P'_2 as P_2 , and to its cardinality as m .

We follow a standard approach, already used in several earlier works, such as [9, 15], and consider the set

$$Q = \{(a, p, b, q) \mid a, b \in P_1, p, q \in P_2, \|a - p\| = \|b - q\|, (a, p) \neq (b, q)\}$$

of quadruples. Enumerate the elements of $\mathcal{D}(P_1, P_2)$ as d_1, d_2, \dots, d_x , and define

$$E_i = \{(a, p) \mid a \in P_1, b \in P_2, \|a - p\| = d_i\},$$

for $i = 1, \dots, x$. Then, by the Cauchy-Schwarz inequality, it follows that

$$|Q| = 2 \sum_{i=1}^x \binom{|E_i|}{2} \geq \sum_{i=1}^x (|E_i| - 1)^2 \geq \frac{1}{x} \left(\sum_{i=1}^x (|E_i| - 1) \right)^2 = \frac{(mn - x)^2}{x}.$$

We may assume $x \leq \frac{mn}{2}$, for otherwise we are done. Then

$$x \geq \frac{m^2 n^2}{4|Q|}. \quad (1)$$

To obtain the asserted lower bound on x , we now derive an upper bound for $|Q|$. For that, we employ a result due to Agarwal et al. [1]. In the language of [1], a family C of *pseudo-parabolas* is a family of graphs of everywhere defined continuous functions, so that each pair intersect in at most two points. The collection C admits a *3-parameter algebraic representation* if it is invariant to translation, its elements are all semi-algebraic with constant description complexity, and have *three degrees of freedom*, in the sense that each curve in C can be specified by three real parameters, and is thus identifiable with a point in \mathbb{R}^3 , in a suitable manner. A full definition of this property is given at the beginning of [1, Section 5].

Theorem 2 (Agarwal et al., [1, Theorem 6.6]) *Let C be a finite family of distinct pseudo-parabolas that admits a 3-parameter algebraic representation, and let P be a finite set of distinct points in the plane. Then*

$$I(P, C) \in O\left(|P|^{2/3}|C|^{2/3} + |P|^{6/11}|C|^{9/11} \log^{2/11} |C| + |P| + |C|\right).$$

(The bound in [1] is slightly weaker; the improvement, manifested in the factor $\log^{2/11} |C|$, which replaces a slightly larger factor in [1], is due to Marcus and Tardos [11].)

Define $Q_0 = \{(a, p, b, q) \in Q \mid p = q\}$ and $Q_1 = Q \setminus Q_0$. For each $a \in P_1$, $p \in P_2$, there can be at most one point $b \neq a$ in l such that $|ap| = |bp|$, since a circle around p intersects l in at most two points. Thus,

$$|Q_0| \in O(nm). \quad (2)$$

We provide an upper bound for $|Q_1|$ by reducing the problem to a curve-point incidence problem and then applying Theorem 2. Combining this with (2) yields an upper bound for $|Q|$.

Let $a = (x, 0), b = (y, 0) \in P_1$, and $p = (p_1, p_2), q = (q_1, q_2) \in P_2$, such that $(a, p, b, q) \in Q_1$. Then

$$\sqrt{(x - p_1)^2 + p_2^2} = \sqrt{(y - q_1)^2 + q_2^2}, \quad (3)$$

or

$$x^2 - y^2 - 2p_1x + 2q_1y + p_1^2 - q_1^2 + p_2^2 - q_2^2 = 0. \quad (4)$$

For fixed p, q , equation (4) defines a hyperbola h_{pq} . Define $\Gamma = \{h_{pq} \mid p \neq q \in P_2\}$. We claim that the hyperbolas of Γ are all distinct. Indeed, let $(p, q) \neq (p', q')$ be two distinct pairs of points from P_2 . Then either $p \neq p'$ or $q \neq q'$, and, from the assumption that each point in P_2 has a distinct x -coordinate, either the coefficients of x or the coefficients of y

in h_{pq} and $h_{p'q'}$ must differ, from which it easily follows that the two hyperbolas are indeed distinct.

Define $\Pi = \{(x, y) \mid (x, 0), (y, 0) \in P_1\}$. A quadruple $(a = (a_x, 0), p, b = (b_x, 0), q)$ is in Q_1 if and only if $(a_x, b_x) \in \Pi$ is incident to $h_{pq} \in \Gamma$, and thus

$$|Q_1| = I(\Pi, \Gamma). \quad (5)$$

Two distinct hyperbolas may share a common component only if they are both degenerate (i.e., the product of two lines). Fortunately, we have the following property.

Lemma 3 *There are no degenerate hyperbolas in Γ .*

Proof. By (3), the hyperbola h_{pq} can be written as

$$(x - p_1)^2 - (y - q_1)^2 = q_2^2 - p_2^2,$$

which can degenerate into two lines if and only if $q_2^2 = p_2^2$. Since we assume that the points of P_2 have distinct y -coordinates, all of the same sign, we have $q_2^2 \neq p_2^2$ (recall that we only consider here hyperbolas h_{pq} with $p \neq q$), and the lemma follows. \square

All hyperbolas of Γ are contained in the family H of hyperbolas whose equations are of the form $(x + \alpha)^2 - (y + \beta)^2 + \gamma = 0$, where γ is either positive or negative (but not zero). Note that half of the hyperbolas of Γ satisfy $\gamma > 0$ and the other half satisfy $\gamma < 0$ (one is obtained from the other by interchanging p and q). The treatments of these two subclasses of hyperbolas are essentially identical, and we focus here on those hyperbolas with $\gamma > 0$, and continue to denote this set as Γ . Each of these hyperbolas has a top branch and a bottom branch. We partition each hyperbola in H into its two branches, and denote by H_T (resp., H_B) the set of the top (resp., bottom) branches of the hyperbolas in H . Each branch is a semialgebraic set of degree 2, and the graph of a continuous totally defined function. Moreover, as is easily seen, each pair of elements of H_T or of H_B intersect in at most two points, and these families are invariant under translation. Finally, each of the branches in H_T or in H_B can be specified by the three real parameters α, β, γ . Hence, H_T and H_B meet the conditions in Theorem 2. Let Γ_T (resp., Γ_B) denote the set of the top (resp., bottom) branches of the hyperbolas of Γ . We have $\Gamma_T \subseteq H_T$ and $\Gamma_B \subseteq H_B$. Combining Theorem 2 with (5), we have

$$I(\Pi, \Gamma_L), I(\Pi, \Gamma_R) \in O\left(|\Pi|^{2/3} |\Gamma|^{2/3} + |\Pi|^{6/11} |\Gamma|^{9/11} \log^{2/11} |\Gamma| + |\Pi| + |\Gamma|\right), \quad \text{and thus}$$

$$|Q_1| = I(\Pi, \Gamma_L) + I(\Pi, \Gamma_R) \in O\left(n^{4/3} m^{4/3} + n^{12/11} m^{18/11} \log^{2/11} m + n^2 + m^2\right).$$

Since this bound subsumes the bound for $|Q_0|$ in (2), it is also an upper bound for $|Q|$. Combining this bound with (1), we then get

$$x \in \begin{cases} \Omega(m^2) & \text{if } m \leq n^{1/2} \\ \Omega(n^{2/3} m^{2/3}) & \text{if } n^{1/2} < m \leq n^{4/5} / \log^{3/5} n \\ \Omega\left(\frac{n^{10/11} m^{4/11}}{\log^{2/11} m}\right) & \text{if } n^{4/5} / \log^{3/5} n < m \leq n^3 \\ \Omega(n^2) & \text{if } n^3 < m, \end{cases}$$

thereby completing the proof.

2.2 Distinct distances in higher dimensions

In this subsection we establish the general version of Theorem 1, in higher dimensions, by modifying the proof for the planar case in a fairly straightforward manner. Let $k > 2$ be some fixed dimension, let l be a line in \mathbb{R}^k , which, without loss of generality, we take to be the x_1 -axis. Let P_1 be a set of n points on l , and let P_2 be an arbitrary set of m points in \mathbb{R}^k , such that no hyperplane orthogonal to l , and no circular hypercylinder whose axis is l contains more than c points of P_2 , for some fixed constant parameter c . That is,

$$\forall t \in \mathbb{R}, \quad |\{p \in P_2 \mid p_1 = t\}| \leq c, \quad \text{and} \quad (6)$$

$$\forall t \in \mathbb{R}, \quad |\{p \in P_2 \mid \rho(p, l) = t\}| \leq c, \quad (7)$$

Notice that this extends the assumption made in the planar case, that no line parallel or orthogonal to l contains more than c points of P_2 .

As in Subsection 2.1, we can assume that P_2 satisfies (6) and (7) with $c = 1$. This is achieved by the same greedy construction, repeatedly choosing an arbitrary point p from P_2 , and then discarding all the at most $2(c-1)$ other points that lie in the same hyperplane orthogonal to l or in the same circular hypercylinder with l as an axis. We get a subset P'_2 of P with at least $\lfloor m/(2c-1) \rfloor = \Theta(m)$ points with the desired property. Continue to denote P'_2 as P_2 and its size as m .

The proof proceeds along the same lines as the previous proof. We consider the set Q of quadruples, partition it into the subsets Q_0 and Q_1 , where the quadruples (a, p, b, q) in Q_0 (resp., in Q_1) satisfy $p = q$ (resp., $p \neq q$). Since a hypersphere intersects l in at most two points, we again have

$$|Q_0| \in O(nm). \quad (8)$$

For each quadruple $(a, p, b, q) \in Q_1$, we have

$$\sqrt{(a_1 - p_1)^2 + \sum_{i=2}^k p_i^2} = \sqrt{(b_1 - q_1)^2 + \sum_{i=2}^k q_i^2},$$

or

$$a_1^2 - b_1^2 - 2p_1 a_1 + 2q_1 b_1 + \left(\sum_{i=1}^k p_i^2 - \sum_{i=1}^k q_i^2 \right) = 0. \quad (9)$$

As before, for a given pair (p, q) , the set of pairs of x_1 -coordinates of all a, b that satisfy (9) is a hyperbola, defined by the equation

$$(x - p_1)^2 - (y - q_1)^2 + \left(\sum_{i=2}^k p_i^2 - \sum_{i=2}^k q_i^2 \right) = 0. \quad (10)$$

As before, we denote this hyperbola as h_{pq} . Arguing as in the planar case (using (6) with $c = 1$), we conclude that two distinct pairs of points of P_2^2 cannot define the same hyperbola. As in Section 2.1, a quadruple (a, p, b, q) is in Q_1 if and only if the point (a_1, b_1) is incident to the hyperbola h_{pq} , and all these hyperbolas are non-degenerate (because of (7) with $c = 1$). Thus,

$$|Q_1| = |\mathcal{I}(\Pi, \Gamma)|. \quad (11)$$

Γ is contained in the same family of hyperbolas defined in the last section, namely hyperbolas that can be defined by $(x + \alpha)^2 - (y + \beta)^2 + \gamma = 0$. Restricting our attention to hyperbolas that have top and bottom branches (those with $\gamma > 0$), and treating the sets Γ_T, Γ_B of top branches and bottom branches of these hyperbolas separately, the same analysis then yields

$$I(\Pi, \Gamma_L), I(\Pi, \Gamma_R) \in O\left(|\Pi|^{2/3} |\Gamma|^{2/3} + |\Pi|^{6/11} |\Gamma|^{9/11} \log^{2/11}(|\Gamma|) + |\Pi| + |\Gamma|\right), \quad \text{and thus}$$

$$|Q_1| = I(\Pi, \Gamma_L) + I(\Pi, \Gamma_R) \in O\left(n^{4/3} m^{4/3} + n^{12/11} m^{18/11} \log^{2/11} m + n^2 + m^2\right).$$

Since this bound subsumes the bound for $|Q_0|$ obtained in (8), it is also an upper bound for $|Q|$, and so

$$|Q| \in O\left(n^{4/3} m^{4/3} + n^{12/11} m^{18/11} \log^{2/11} m + n^2 + m^2\right).$$

Then, by combining this bound with (1), we get

$$x \in \begin{cases} \Omega(m^2) & \text{if } m \leq n^{1/2} \\ \Omega(n^{2/3} m^{2/3}) & \text{if } n^{1/2} < m \leq n^{4/5} / \log^{3/5} n \\ \Omega\left(\frac{n^{10/11} m^{4/11}}{\log^{2/11} m}\right) & \text{if } n^{4/5} / \log^{3/5} n < m \leq n^3 \\ \Omega(n^2) & \text{if } n^3 < m, \end{cases}$$

completing the proof of the theorem. \square

Remarks. An obvious open problem is to extend the result to the case where P_1 lies on some constant-degree algebraic curve, and P_2 is arbitrary. This is probably doable, except that the hyperbolas h_{pq} are now replaced by more general algebraic curves, which no longer satisfy the pseudo-parabolas properties. We leave it as a topic for further research to explore such an extension.

In addition, the requirement that every hyperplane or hypercylinder of the above kinds contains at most $O(1)$ points of P_2 is perhaps rather restrictive. What happens if this number is constrained to be at most s , for some non-constant $s \ll |P_2|$? It would be interesting to find a sharp dependence of the number of distinct distances on s .

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